

LESSON 12

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LESSON PREVIEW

Another type of mathematical notation with vertical components is studied. This lesson also applies the "mathematical statement" format to formal proofs.

Some of the print examples are enlarged in order to show the modifiers more clearly on the printed page. You may also wish to use a magnifier to remove any uncertainty.

MODIFIERS AND MODIFIED EXPRESSIONS**[NC Rule 15]**

Some mathematical notation has a vertical aspect that challenges linear braille notation. By using modified expression indicators, the transcriber can relay the material in a compact manner, and the reader can construct the symbols into a meaningful expression.

12.1 Definition

A modifier is a symbol or a combination of symbols occurring *directly over* or *directly under* its related symbol or expression. Here are some typical samples.

$\overset{?}{=}$	a question mark over an equals sign
3. <u>15</u>	an underlined digit
1. $\overline{37}$	a line over two numerals (signifying a repeating decimal)
\overleftrightarrow{AB}	an arrow over two letters (signifying a line)
\hat{k}	a caret over a letter

12.2 Construction of Simple Modified Expressions – The Five-Step Rule

An expression modified using the five-step method is initiated and terminated with special indicators.

⠠	Multipurpose Indicator
⠨	Termination Indicator

The position of the modifier (above or below the expression) is also shown with the use of an indicator.

⠠	Directly-Over Indicator
⠨	Directly-Under Indicator

The process of constructing a modified expression is known as "The Five-Step Rule".

- (1) The *multipurpose indicator* is placed immediately before the expression to be modified.
- (2) The *expression* to be modified is written second.

PRACTICE 12A

Instructions: Review arrow construction in Lesson 9. In the **Vector Addition** sentence assume that all vectors in the document are shown using that particular arrow notation. Show the proper way to omit the vector arrows in the transcription. Include the required transcriber's note after the topic heading.

Here are two modified arrows: $x \xrightarrow{g} y \xrightarrow{f} z$

Arrows as Modifiers

1. \overleftarrow{F}
2. \overleftarrow{AB}
3. \overleftarrow{CD}
4. $\overrightarrow{OB} \cup \overrightarrow{OC}$
5. $\overset{\circ}{\overrightarrow{EF}}$
6. \overleftarrow{T}
7. $\overleftrightarrow{XZ} \parallel \overleftrightarrow{RS}$
8. $\overleftrightarrow{AB} + \overleftrightarrow{CD}$

Vector Addition

\mathbb{R} equals \overrightarrow{OP} equals \overrightarrow{OM} plus \overrightarrow{MC} plus \overrightarrow{CP} .

PRACTICE 12B

Instructions: Treat the three examples of unit vectors in problem #2 as displayed mathematical material.

Carets and Bars as Modifiers

- Unit vectors can be denoted with normal vector notation, \mathbf{u} or \vec{u} , or with standard unit vector notation $\hat{\mathbf{u}}$, spoken "u-hat".
- Unit vectors in various coordinate systems use Greek and English letters.

Cartesian coordinate system: $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$

Cylindrical coordinate system: $\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}}$

Spherical coordinate system: $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$

3. $1.142857\overline{142857}$

4. $\frac{7}{15} = .\overline{46}$

5. $2 \cdot 3 = \overline{2} \cdot 3 = \overline{2 \cdot 3}$

6. $\overline{PQ}, \overline{x'}, \overline{R''S''}$

7. $\overline{\mathbf{s}}, \overline{\alpha}, \overline{m'}$

8. $m\overline{BC} = a$

9. $\overline{C} = 100 \times 1000$

10. $F = 2\pi\overline{r}l$

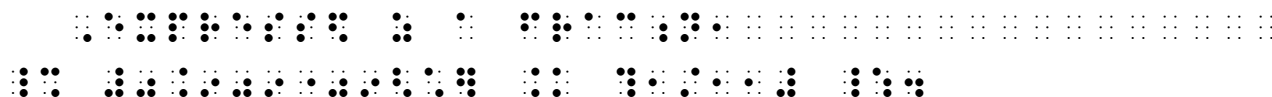
11. $P(\overline{a + bi}) = \overline{0} = 0$

12. $\hat{x}_i = 0.5(\overline{x}_i + \underline{x}_i)$

13. $3.141\overline{59}$

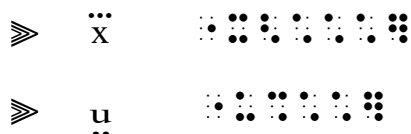
Example 12-12

Expressed as a fraction, $0.909\dot{0}\dot{9} = \frac{1}{11}$.



In print, a single dot is shown over each of the last two digits (0 and 9).

When one or more dots occur over or under a single letter or numeral, the symbol for the dot is used as many times as necessary to conform with the print text.

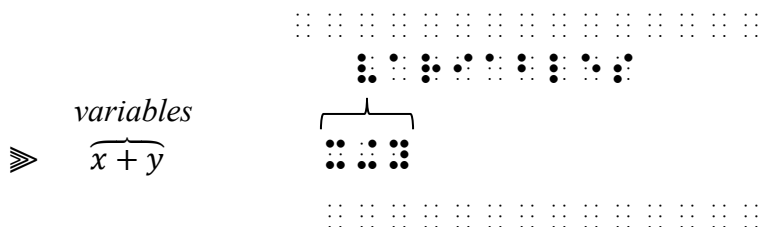


Example 12-13

Prove $\ddot{x} = \frac{d^2x}{dt^2}$

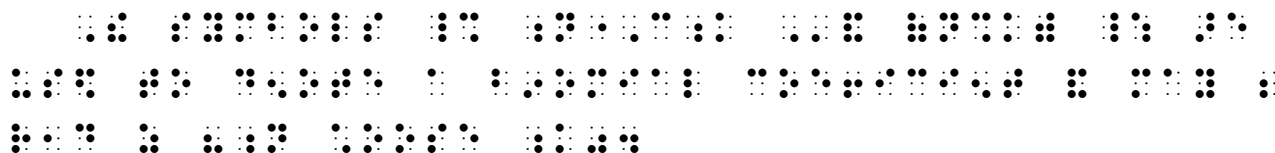


- 12.6.3 **Horizontal Grouping Sign.** When a horizontal grouping sign occurs over or under a mathematical expression, it is either a part of the expression or is pointing to a label.
- As a Pointer.** When the grouping sign points to a label or to explanatory text, it must be drawn as a tactile graphic. Refer to *Guidelines and Standards for Tactile Graphics* for drawing techniques.



Example 12-17

The symbols ${}_nC_k$ and $\binom{n}{k}$ are used to denote a binomial coefficient and may be read as "n choose k".

**PRACTICE 12C**

1. In Figure 7.3, if $\widehat{AB} = \widehat{CD}$ in circle O, then $\angle AOB = \angle BOC$.
2. $x \stackrel{?}{=} y$ means "does x equal y ?"
3. $\sum_{d|n}$ (where $d|n$ means " d divides n ").
4. $\binom{t}{p} = R_t^p$
5. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for all integers $n \geq 0$.
6. Does (\tilde{x}, \tilde{y}) mean \tilde{x} and \tilde{y} ?
7. More modified expressions:

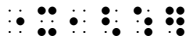
a. $.249\dot{9}$	b. $2.431\dot{3}\dot{1}$	c. $\dot{x}\dot{y} - \dot{y}\dot{x}$
d. $\underline{\underline{a}} + \underline{\underline{a}} = ?$	e. $f \rightarrow \tilde{f}$	f. $\widehat{x + y}$

$$\gg \overline{OD^2} + \overline{OP^2}$$



The first dot 5 begins the first modified expression. The second dot 5 is a baseline indicator following the first superscript "2". Similarly, the third dot 5 begins the second modified expression and the fourth dot 5 is a baseline indicator following the second superscript.

$$\gg \overline{x_1}$$



You may wish to review Section 6.11 in Lesson 6 regarding nonuse of the subscript indicator.

- a. **Binomial Coefficient.** Although a binomial coefficient is not technically a modified expression, notice how this rule applies.

$$\gg \begin{pmatrix} a_x \\ b_y \end{pmatrix}$$



The baseline indicator precedes the "directly under" indicator. This keeps that indicator on the same level of writing as the letters "a" and "b".

- 12.9.3 **Modifier Ends with a Superscript or Subscript.** The terminator must be on the same level as the multipurpose indicator which starts the modified expression. When the last character(s) in the modifier is at the superscript or subscript level and a level indicator has been used, the baseline indicator precedes the terminator.

$$\gg \text{median}_{(s, t) \in S_{xy}}$$



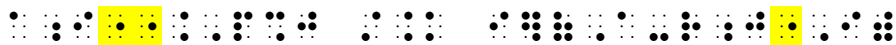
The first dot 5 begins the modification which is on the baseline of writing. The second dot 5 is a baseline indicator, which puts the terminator on the same level as the modified expression. (The membership symbol was introduced in Lesson 5.)

PRACTICE 12E**Superscripts and Subscripts**

- A) $\overline{AB}^2 + \overline{BC}^2$
- B) $\overline{A} = [\overline{a}_i]$
- C) $\sqrt{\dot{x}^2 + \dot{y}^2}$
- D) \underline{Z}°
- E) If $\overline{a}_1 = 72$, find \overline{a}_7 .
- F) Draw $\overline{P_1P_2}$ if P_1 is the point $(1, 3)$ and P_2 is the point $(2, -1)$.
- G) $(\overline{3^{-1}}) \in P$
- H) $\overline{x}_1 + \overline{y}_1$
- I) $\lim_{x \rightarrow 1^+}$
-

immediately followed by a dot 5 beginning the modified expression. Finally, the directly-over indicator is preceded by a baseline indicator to ensure that the entire modified expression is transcribed on the same level.

$$\gg a_i \prod_{j \neq i} (A - r_j I)$$



Because a_i requires a subscript indicator and the five-step rule is used for the modified expression, two dot 5s are needed. The baseline indicator is immediately followed by a dot 5 beginning the modified expression. The last dot 5 is a baseline indicator following the subscript "j".

12.10 Modified Expression Within a Superscript or Subscript

A modifier and its related expression must be placed on the same level of writing. When a modified expression occurs as a superscript or subscript, or as the first part of a superscript or subscript, the level indicator is transcribed first, followed by the multipurpose indicator which begins the modified expression. This ensures that the expression as a whole appears on the same level of writing.

$$\gg S^{\tilde{x}}$$

This dot 5 begins the modified expression \tilde{x} (which is in the superscript position). It will not be misread as a baseline indicator because nothing comes between it and the superscript indicator.

$$\gg S_{\tilde{x}}$$

This dot 5 begins the modified expression \tilde{x} (which is in the subscript position). It will not be misread as a baseline indicator because nothing comes between it and the subscript indicator.

$$\gg \tilde{x}A_1$$

The first dot 5 begins the modified expression \tilde{x} (which is a left subscript to the letter A). The second dot 5 is a baseline indicator.

If the modified expression occurs in the middle or at the end of the superscript or subscript, the appropriate level indicator must be repeated before the multipurpose indicator to show continuation of the same level of writing. This ensures that the multipurpose indicator will not be misread as a baseline indicator.

Modified Signs of Comparison

12.11 Definition

A modified sign of comparison consists of a simple sign of comparison, such as the equals sign or the tilde, modified by a caret, dot, triangle, question mark, vertical bar, or any symbol except another sign of comparison.

When a simple sign of comparison occurs above or below another simple sign of comparison the combination is transcribed as a sign of comparison compounded vertically. See Section 5.8 in Lesson 5 for a review of that construction. Note that many of those signs are printed with a single horizontal line "bar over" or "bar under". For example, "bar over greater than, inclusion with bar under, bar over single tilde, logical sum with bar under," etc. These signs are not to be misinterpreted as a horizontal bar modifying a sign of comparison.

12.12 Transcription

A modified sign of comparison is transcribed in accordance with the five-step rule for modified expressions.






In addition to the caret and inverted caret seen earlier in this lesson, you may also encounter a left- or right-pointing caret in a modified sign of comparison. Do not confuse these two symbols with the "less than" and "greater than" comparison signs. Ask an expert if context does not clarify the identity of this symbol.

⠠⠠⠠⠠⠠⠠⠠⠠	Left-Pointing Caret	<
⠠⠠⠠⠠⠠⠠⠠⠠	Right-Pointing Caret	>

The following list contains the modified equals signs most commonly used.

Modified Equals Sign		
⠠⠠⠠⠠⠠⠠⠠⠠	Caret Over Equals Sign	⠠⠠
⠠⠠⠠⠠⠠⠠⠠⠠	Caret Under Equals Sign	⠠⠠ ⠠
	("is projective to")	
⠠⠠⠠⠠⠠⠠⠠⠠	Inverted Caret Over Equals Sign	⠠⠠
⠠⠠⠠⠠⠠⠠⠠⠠	Left-Pointing Caret Over Equals Sign	⠠⠠
⠠⠠⠠⠠⠠⠠⠠⠠	Right-Pointing Caret Over Equals Sign	⠠⠠

The remainder of the list provided in the code contains other modified comparison signs most commonly used.

Modified Tilde		
	Dot Under Tilde	\sim
	Dot Over Tilde	$\overset{\cdot}{\sim}$
Modified Horizontal Bar		
	Caret Over Bar	$\overset{\wedge}{-}$
	Caret Under Bar	$\underset{\wedge}{-}$
	("is perspective to")	
	Dot under Bar	$\overset{\cdot}{-}$

If the horizontal bar is modified by a dot *over* it, the combination is a modified sign of operation ("minus with dot over" signifying "proper difference"). The five-step rule is not used for this symbol. See Section 5.4.7 in Lesson 5.

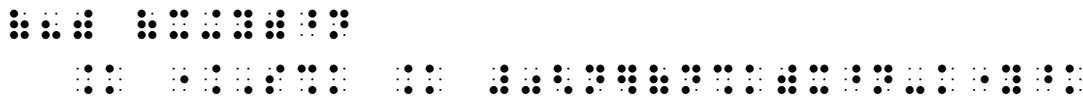
PRACTICE 12G

Instructions: Use the five-step rule to show the horizontal grouping sign in the last item.

Modified Signs of Comparison and More

- $A \doteq 3.14r^2$
- $\angle b \doteq \frac{1}{2} \widehat{EB}$
- The symbol $\overset{\Delta}{=}$ is used to make a definition.
- $x \sim \mathcal{N}(0, 1)$
- $x^n = \underbrace{x \cdot x \cdot x \cdot \dots \cdot x}$

$$(8) \quad (x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$



This is a linked expression where $(x + y)^n$ is the anchor and

$= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$ is the link. When a linked expression will not fit on one

braille line, start a runover line with the link. (Lesson 8) The equals sign within the modification is not a suitable division site because the components of an expression modified according to the five-step rule should not be divided between lines. (See [Section 12.2.](#))

PRACTICE 12H

Expressions with More Than One Modifier

$$1. \overline{\overline{9 \cdot 3}} = \overline{\overline{9 \cdot 3}}$$

$$5. \overline{\overline{x_a + y^n}}$$

$$2. \overline{\overline{A^n}}$$

$$6. \overline{\overline{N}}$$

$$3. \overline{\overline{A \cap B \cap C}}$$

$$7. \sum_{i=1}^k \sum_{j=1}^k$$

$$4. \overline{\overline{a_n + b_p}}$$

$$8. \prod_{\substack{j=1 \\ j \neq k}}^n$$

Format

12.16 Formal Proof [NC Rule 26.7]

A proof is a valid argument that establishes the truth of a mathematical statement. It is often introduced by a heading such as *Theorem*, *Proposition*, or *Lemma*. A progressive sequence of statements leads to the conclusion. In a formal proof, every step of the argument is shown, and each step is supported by a definition or by a previously proven statement.

Lesson 11 explained how to transcribe a mathematical statement. Those guidelines are summarized in items a-c, below, in the context of a proof.

- a. A blank line precedes the beginning of the proof.
- b. *Heading*: The heading can be formatted as a paragraph heading or as a cell-5 or cell-7 heading, at the transcriber's discretion. Review Section 11.38.b in Lesson 11 for details.
- c. *Statement*: Continue with the text, using normal (3-1) paragraph style. When the statement is printed in a variant typeform and the proof follows, in regular type, it is recommended that the typeform be preserved for the statement in order to retain distinction.
- d. *Auxiliary Captions*: Paragraph headings such as Given, Hypothesis, Prove, or Conclusion begin in cell 3, without a blank line before the paragraph. Associated material follows the caption. Runovers go in cell 1.
- e. *Two-Column Proof*: See [Section 12.16.1](#).
- f. *End of Proof Icon*: See [Section 12.16.2](#).
- g. When the proof is complete, insert a blank line before continuing with the text.

12.16.1 **Two-Column Proof.** When a formal proof is presented by numbered steps printed in two columns, the layout is changed as follows.

- a. If there is a caption such as "Proof", follow the same pattern established in [Section 12.16.d](#), "Auxiliary Captions".
- b. The column format is changed to a list in braille. A transcriber's note must call attention to the change in format. See [Section 12.16.1.c](#) for a sample transcriber's note.

A blank line is inserted before the list. If there are column headings, such as "Statement" and "Reason", see [Section 12.16.1.c](#). Each step begins in cell 1, starting with the first item from the left column. Runovers are in cell 3. The related item from the right column begins in cell 1 on the next line, with runovers in cell 3.

- c. *Identifiers*: Each item must be labeled with an identifier. Typically, the print copy includes the column headings "Statements" and "Reasons". In braille, the column headings are replaced with a letter – "S" or "R", respectively – as part of each item number. For example, "1S" is Statement 1, followed on the next line by "1R" which is Reason 1. If other

Example 12-21

(Assume the required transcriber's note regarding the two-column proof appears on the Transcriber's Notes page.)

THEOREM 2. All right angles are equal.

Given: $\angle ABC$ and $\angle DEF$ are right angles.

Prove: $\angle ABC$ equals $\angle DEF$.

<i>Statements</i>	<i>Reasons</i>
1. $\angle ABC$ and $\angle DEF$ are right angles.	1. Given.
2. $\angle ABC = 90^\circ$, $\angle DEF = 90^\circ$.	2. A right angle contains 90 degrees.
3. $\angle ABC = \angle DEF$.	3. Transitivity postulate. ■

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Line 1: In this course, we do not allow a mathematical statement to start on line 1. (See Section 11.38.a in Lesson 11.) Box lines are not required.

Line 2: The paragraph heading is printed in capital letters and also in a nonregular typeface (boldface). The capitals are retained and typeform (boldface) is disregarded.

Lines 2-3: Typeform (bold) is preserved for the statement in order to retain distinction. See [Section 12.16.c](#).

Lines 4-7: Each auxiliary caption follows print regarding typeform (italics, in this example), and uses a 3-1 paragraph style.

Line 8: A blank line precedes the list.

Lines 9-16: Each item in the 2-column proof begins in cell 1, with runovers in cell 3. Identifying letters S and R are combined with each step number.

Line 16: A dark square is printed in the right margin to mark the end of the proof. The "qed" icon is transcribed.

Line 17: A blank line follows the proof.

PRACTICE 12I

Instructions: Create a Transcriber's Notes page that would appear in a volume which contains the proof shown in Practice 12J. Remember to include the first paragraph citing the Nemeth code. In the second paragraph, explain the step-number format as described in [Section 12.16.1.c](#).

PRACTICE 12J

Explain the properties given as reasons in this proof.

Given: $3x = 7 - \frac{1}{2}x$

To Prove: $x = 2$

STEP	REASON
1. $3x = 7 - \frac{1}{2}x$	1. GIVEN
2. $6x = 14 - x$	2. Multiplication Property
3. $7x = 14$	3. Addition Property
4. $x = 2$	4. Division Property

For further practice, see Addendum 1—Reading Practice.

Submit Exercise 12 to your instructor.

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